Int. J. Solids Structures Vol. 29, No. 17, pp. 2137–2157, 1992
Printed in Great Britain.

0020-7683/92 $5.00+ .00
© 1992 Pergamon Press Ltd

FINITE ELASTOPLASTIC TRANSFORMATIONS OF TRANSVERSELY ISOTROPIC METALS

N. ARAVAS
Department of Mechanical Engineering and Applied Mechanics, University of Pennsylvania, Philadelphia, PA 19104, U.S.A.

(*Received* 18 *August* 1991 ; *in revised form* 11 *December* 1991)

Abstract—A finite strain theory for transversely isotropic elastic–plastic materials is developed. The formulation is based on a multiplicative decomposition of the deformation gradient tensor into elastic and plastic parts. The axis of transverse isotropy at each material point is assumed to "follow" the deformation of the continuum. We derive a constitutive equation for the plastic spin $\mathbf{W}^p_r$, which is the average spin of the continuum as seen by an observer spinning with the substructure (e.g. the fibers in a metal–matrix composite). It is shown that $\mathbf{W}^p_r = \mathbf{mm}\cdot\mathbf{D}^p_r - \mathbf{D}^p_r\cdot\mathbf{mm}$, where $\mathbf{D}^p_r$ is the plastic part of the deformation rate, and $\mathbf{m}$ is the unit vector in the direction of transverse isotropy in the intermediate (isoclinic) configuration. The numerical implementation of the developed model in a finite element program as well as an algorithm for the numerical integration of the elastoplastic equations are discussed in detail. The problem of plane strain extrusion of a metal–matrix composite reinforced by short aligned fibers is solved using the finite element method.

1. INTRODUCTION

The mechanics of finite elastoplastic deformations has been well developed in recent years and the problem of the appropriate generalization of the classical laws of elastoplasticity to the case of finite deformations has been addressed in numerous publications. We mention amongst these the work of Hill (1966, 1967), Lee (1969), Mandel (1971a), Hill and Rice (1972), Asaro and Rice (1977), Dafalias (1983, 1985a,b, 1987a, 1988) and Loret (1983). Significant progress has also been made in the development of new algorithms for the numerical integration of the elastoplastic constitutive equations in the presence of finite strains and rotations (Nagtegaal and De Jong, 1981 ; Simo and Ortiz, 1985; Simo, 1985; Moran *et al.*, 1990; Weber and Anand, 1990).

A detailed analysis of the mechanical behavior of transversely isotropic elastoplastic solids under finite isothermal deformations is presented in this paper. We consider an elastoplastic material which is characterized by *persistent* transversely isotropic symmetries in its relaxed (elastically unloaded) configuration. At each material point in the undeformed configuration $\mathfrak{B}_0$ we specify a material direction that defines the *local* axis of transverse isotropy. The local axis of symmetry is embedded in the continuum and follows its deformation. To be more precise, we let $\mathrm{d}\mathbf{X}$ be an infinitesimal material line element emanating from a material point A at $\mathfrak{B}_0$ along the local axis of symmetry; when the material is subject to finite plastic strains, the local direction of transverse isotropy is defined by the new orientation of the material element $\mathrm{d}\mathbf{X}$, namely $\mathbf{F}^p\cdot\mathrm{d}\mathbf{X}$, where $\mathbf{F}^p$ is the plastic part of the deformation gradient that defines the intermediate unstressed configuration as discussed in detail in the following section.

As an example, we consider the geometry shown in Fig. 1, where an elastoplastic beam refinforced by short fibers is bent plastically. We denote by $\mathfrak{B}_0$ and $\mathfrak{B}$ the initial and final configurations respectively. Let $\mathbf{m}_0$ be the unit vector in the direction of the fiber at a

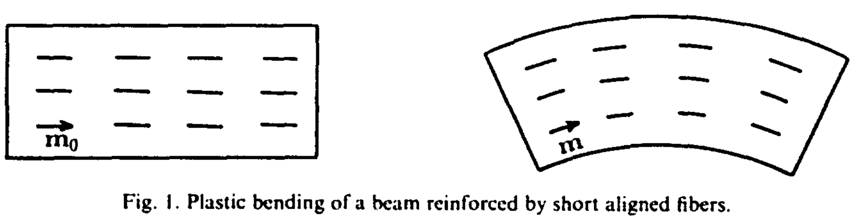

Fig. 1. Plastic bending of a beam reinforced by short aligned fibers.

material point at $\mathfrak{B}_0$. Although in a real fiber-reinforced composite material $\mathbf{m}_0$ exists only at points occupied by fibers, in our continuum model we assume that $\mathbf{m}_0$ is a continuous vector field defined everywhere in the body. After plastic bending, the direction of the corresponding material fiber $\mathfrak{B}$ is defined by the unit vector $\mathbf{m} = \mathbf{F}\cdot\mathbf{m}_0/|\mathbf{F}\cdot\mathbf{m}_0|$, the direction of $\mathbf{m}$ depending on the particular material point considered (see Fig. 1). Assuming that the elastic strains are small, we can identify the configuration $\mathfrak{B}$ (to within elastic strains) with the aforementioned relaxed (unstressed) configuration where the material symmetries are defined. Our interpretation of persistent transverse isotropy is that the beam at $\mathfrak{B}$ is *locally* transversely isotropic with the symmetry axis defined locally by the direction of $\mathbf{m}$. The idea of the axis of transverse isotropy being convected with the deformation was first incorporated in plasticity theories by Mulhern *et al.* (1967, 1969).

The spin of the characteristic material direction (e.g. fiber in a composite material) is, in general, different from the average spin of the continuum. The quantity that emerges from such a distinction in kinematics is the plastic spin, which is the average spin of the continuum *relative* to the material substructure. Mandel (1971a, 1973) and Kratochvil (1971, 1973) were the first to suggest that a complete macroscopic elastoplasticity theory must include constitutive relations not only for the plastic part of the deformation rate but for the plastic spin as well. Using the representation theorems for isotropic functions, Kratochvil (1973) concluded that the plastic spin vanishes identically in isotropic materials. In anisotropic materials, however, the plastic spin is of major importance and has been the focus of a series of papers by Dafalias (1983, 1984, 1985a,b, 1987a, 1988) and Loret (1983) where constitutive equations are formulated for different anisotropies using tensorial structure variables.

In Section 2 we discuss the kinematics of finite elastoplastic transformations and identify the intermediate unstressed configuration with Mandel's (1971a) isoclinic configuration. The elastoplastic constitutive equations are presented in Section 4 and a constitutive equation for the plastic spin is developed. The numerical implementation of the developed model in a finite element program and an algorithm for the numerical integration of the elastoplastic equations are presented in Sections 5 and 6. The problems of finite simple shear and plain strain extrusion of a fiber-reinforced metal-matrix composite are solved.

Standard notation is used throughout. Boldface symbols denote tensors the orders of which are indicated by the context. The prefices tr and det indicate the trace and the determinant respectively, a superscript T the transpose, subscripts s and a the symmetric and anti-symmetric parts of a tensor, and a superposed dot the material time derivative. All tensor components are written with respect to a fixed Cartesian coordinate system. Let $\mathbf{a}$ and $\mathbf{b}$ be vectors, $\mathbf{A}$ and $\mathbf{B}$ second order tensors, and $\mathbf{C}$ and $\mathbf{D}$ fourth order tensors; the following products are used in the text $(\mathbf{ab})_{ij} = a_i b_j$, $(\mathbf{A}\cdot\mathbf{b})_i = A_{ik}b_k$, $(\mathbf{b}\cdot\mathbf{A})_i = b_k A_{ki}$, $(\mathbf{A}\cdot\mathbf{B})_{ij} = A_{ik}B_{kj}$, $(\mathbf{AB})_{ijkl} = A_{ij}B_{kl}$, $\mathbf{A}:\mathbf{B} = A_{ij}B_{ij}$, $(\mathbf{C}:\mathbf{A})_{ij} = C_{ijkl}A_{kl}$, $(\mathbf{A}:\mathbf{C})_{ij} = A_{kl}C_{klij}$, and $(\mathbf{C}:\mathbf{D})_{ijkl} = C_{ijmn}D_{mnkl}$, where the summation convention is used for repeated indices. We also denote by $\mathbf{J}$ the symmetric unit fourth order tensor with Cartesian components $J_{ijkl} = (\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})/2$, δ_{ij} being Kronecker's delta.

2. KINEMATICS

The kinematics of finite elastic plastic deformation is described by the multiplicative decomposition of the deformation gradient $\mathbf{F}$:

$$\mathbf{F} = \mathbf{F}^e\cdot\mathbf{F}^p, \tag{1}$$

formally introduced in continuum mechanics by Lee and Liu (1967) and Lee (1969). According to this decomposition, the neighborhood of a material point is mapped first from the undeformed configuration $\mathfrak{B}_0$ to the intermediate *unstressed* configuration $\mathfrak{B}_i$ by the plastic part $\mathbf{F}^p$, and then carried to the current configuration by the elastic part $\mathbf{F}^e$.

In the case of a strong Bauschinger effect when the elastic region does not include the stress origin, the stress cannot be reduced to zero without causing additional plastic deformation. In such a case, the unstressed configuration is only notional and is reached by a "virtual" elastic unloading with all active or potentially active mechanisms of plastic flow "frozen" (Mandel, 1973; Lee, 1981). A detailed discussion on the definition of the unstressed configuration has been presented by Dafalias (1987b). Here, we *assume* that the stress origin is always inside the yield surface.

We consider a transversely isotropic elastoplastic continuum, in which the local axis of symmetry is defined by the orientation of a certain infinitesimal material fiber **dX** at each point. The multiplicative decomposition (1) is written for each material point and the intermediate configuration $\mathfrak{B}_i$, determined by $\mathbf{F}^p$, is defined in such a way that the orientation of the aforementioned material fiber at $\mathfrak{B}_i$ with respect to a global system is the same as the corresponding orientation in the reference configuration $\mathfrak{B}_0$ (see Fig. 2). The intermediate configuration is now the so-called "isoclinic configuration" introduced by Mandel (1971a, 1974), and is uniquely defined to within a rigid rotation about the local axis of symmetry.

The velocity gradient **L** can be written as

$$\mathbf{L} = \dot{\mathbf{F}} \cdot \mathbf{F}^{-1} = \dot{\mathbf{F}}^e \cdot \mathbf{F}^{e-1} + \mathbf{F}^e \cdot \dot{\mathbf{F}}^p \cdot \mathbf{F}^{p-1} \cdot \mathbf{F}^{e-1}. \tag{2}$$

The deformation rate **D** and the spin **W**, defined as the symmetric and anti-symmetric parts of **L**, are now written as (Willis, 1969; Freund, 1970; Asaro and Rice, 1977; Nemat-Nasser, 1982)

$$\mathbf{D} = \mathbf{D}^e + \mathbf{D}^p, \tag{3}$$

and

$$\mathbf{W} = \mathbf{W}^* + \mathbf{W}^p, \tag{4}$$

where

$$\mathbf{D}^e = (\dot{\mathbf{F}}^e \cdot \mathbf{F}^{e-1})_s, \quad \mathbf{W}^* = (\dot{\mathbf{F}}^e \cdot \mathbf{F}^{e-1})_a, \tag{5}$$

$$\mathbf{D}^p = (\mathbf{F}^e \cdot \dot{\mathbf{F}}^p \cdot \mathbf{F}^{p-1} \cdot \mathbf{F}^{e-1})_s, \quad \mathbf{W}^p = (\mathbf{F}^e \cdot \dot{\mathbf{F}}^p \cdot \mathbf{F}^{p-1} \cdot \mathbf{F}^{e-1})_a. \tag{6}$$

In the isoclinic configuration we also define

$$\mathbf{L}_i^p = \dot{\mathbf{F}}^p \cdot \mathbf{F}^{p-1}, \quad \mathbf{D}_i^p = (\dot{\mathbf{F}}^p \cdot \mathbf{F}^{p-1})_s \quad \text{and} \quad \mathbf{W}_i^p = (\dot{\mathbf{F}}^p \cdot \mathbf{F}^{p-1})_a. \tag{7}$$

We mention that an arbitrary rigid body rotation can be superposed to the isoclinic configuration and still leave the material point under consideration unstressed. In that sense, the intermediate unstressed configuration is not uniquely defined and the isoclinic configuration is just a convenient choice. A thorough discussion on the definition of the

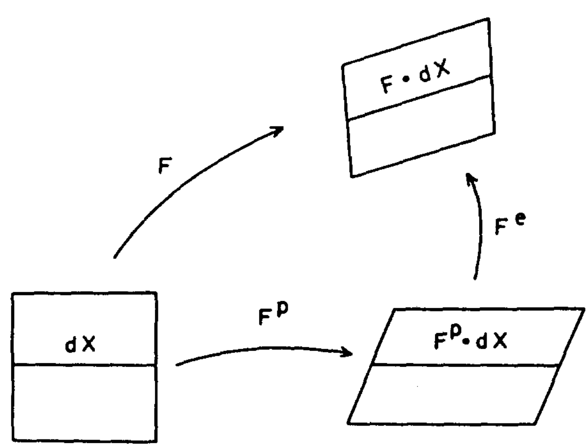

Fig. 2. Schematic representation of the multiplicative decomposition of the deformation gradient.

intermediate configuration has been presented by Dafalias (1987a, 1988), who showed that the choice of that configuration can be made arbitrarily, provided that the corresponding kinematic quantities, such as the elastic and plastic parts of the deformation rate and the plastic spin, are properly defined. It is also important to realize that the spin of the substructure (e.g. fibers in a composite material) is, in general, different from the average spin $\mathbf{W}$ of the continuum, and that the objective rates used in the constitutive equations must be co-rotational with the substructure (as opposed to the continuum) (Mandel, 1971a; Dafalias, 1984). When the constitutive equations are written in the isoclinic configuration, the aforementioned co-rotational rates are simply the usual time derivatives, since the orientation of the substructure does not vary with time in the isoclinic configuration. In this connection, we mention that the definitions of $\mathbf{D}^e$, $\mathbf{D}^p$, $\mathbf{W}^p$ and $\mathbf{W}^*$, given above, are appropriate when the intermediate configuration is isoclinic; when $\mathfrak{B}_i$ is not isoclinic, however, the material time derivatives in (5) and (6) must be replaced by derivatives co-rotational with the substructure (Dafalias, 1987a). A detailed discussion of the proper definition of the elastic and plastic parts of the deformation rate can be also found in Mandel (1971b, 1981).

The constitutive equations can be written in any one of the three configurations $\mathfrak{B}_0$, $\mathfrak{B}_i$, and $\mathfrak{B}$, and then appropriately transformed to any of the other two if desired. We choose to define the material symmetries and write the constitutive equations and the yield condition at the intermediate configuration; this is, in a sense, a natural choice, since $\mathfrak{B}_i$ is the reference (or "undeformed") configuration for the elastic part of the deformation $\mathbf{F}^e$, and the "current" (or deformed) configuration for the plastic part $\mathbf{F}^p$ [see also Willis (1969), Freund (1970), Kratochvil (1973), Fardshisheh and Onat (1974) and Hahn (1974)]. We also choose to work in terms of the isoclinic configuration so that the use of co-rotational rates in the constitutive equations is avoided and the developed model is easier to implement in a finite element program.

3. WORK-CONJUGATE VARIABLES

In this section we briefly discuss several work-conjugate variables defined in the current as well as in the isoclinic configuration. The rate of stress working per unit mass is

$$\dot{W} = \frac{1}{\rho_0}\,\mathrm{tr}\,(\boldsymbol{\tau}\cdot\mathbf{D}), \tag{8}$$

where ρ_0 is the mass density in the undeformed configuration $\mathfrak{B}_0$, $\boldsymbol{\tau} = J\boldsymbol{\sigma}$ is the Kirchhoff stress, $\boldsymbol{\sigma}$ is the true or Cauchy stress defined in the current configuration and $J = \det(\mathbf{F})$.

Using eqns (5)–(7) we derive the following equivalent expressions for $\dot{W}$:

$$\dot{W} = \frac{1}{\rho_0}(\mathrm{tr}\,(\boldsymbol{\tau}\cdot\mathbf{D}^e) + \mathrm{tr}\,(\boldsymbol{\tau}\cdot\mathbf{D}^p)) = \frac{1}{\rho_0}(\mathrm{tr}\,(\mathbf{t}^e\cdot\dot{\mathbf{F}}^e) + \mathrm{tr}\,(\boldsymbol{\Sigma}\cdot\mathbf{L}_i^p)), \tag{9}$$

where $\mathbf{t}^e = J^e\mathbf{F}^{e-1}\cdot\boldsymbol{\sigma}$ is the elastic nominal (first Piola–Kirchhoff) stress tensor defined at $\mathfrak{B}_i$, $J^e = \det(\mathbf{F}^e)$, and

$$\boldsymbol{\Sigma} = \mathbf{F}^{e-1}\cdot\boldsymbol{\tau}\cdot\mathbf{F}^e. \tag{10}$$

We mention in passing that, when the material is plastically incompressible, $J^p = \det(\mathbf{F}^p) = 1$ and $J^e = J$; in the general case, however, where $J^p \neq 1$, we find that $J = J^eJ^p$.

The *non-symmetric* stress tensor $\boldsymbol{\Sigma}$ has been used in the mechanics of finite deformations of single crystals to define the "thermodynamic shear stresses" that govern slip (Rice, 1971) or, equivalently, the "generalized Schmid stress" (Hill and Havner, 1982), and by Hahn (1974) in his finite deformation theoroy of plasticity; the transpose of $\boldsymbol{\Sigma}$ has also been used by Mandel (1971a), Halphen (1975), Teodosiu and Sidoroff (1976), and, more recently, by

Van Der Giessen (1989a,b) and Mohan *et al.* (1991) in their studies of finite elastoplastic deformations.

We mention in passing that the invariants of Σ are the same as those of τ, and that the deviatoric part of Σ is related to the deviatoric part of τ through an equation similar to (10), i.e.

$$\Sigma' = \mathbf{F}^{e-1}\cdot\tau'\cdot\mathbf{F}^e, \tag{11}$$

where a prime denotes the deviatoric part of a tensor. It is important to realize, however, that (11) does not necessarily imply that Σ' is independent of the hydrostatic Kirchhoff stress $p = \tau_{kk}/3$, since $\mathbf{F}^e$ is, in general, a function of p.

It is also interesting to note that Σ is not symmetric, but is not an arbitrary second order tensor either. Lubliner (1986, 1990) mentions that the combination $\mathbf{C}^e\cdot\Sigma$ is a symmetric tensor, and consequently Σ must obey the constraint

$$(\mathbf{C}^e\cdot\Sigma)^{\mathrm{T}} = \mathbf{C}^e\cdot\Sigma, \tag{12}$$

where $\mathbf{C}^e$ is determined by $\mathbf{S}^e$ and hence by Σ. The above condition is equivalent to three non-linear equations involving the components of Σ, limiting Σ to a six-dimensional manifold in the nine-dimensional space of second order tensors (Lubliner, 1986, 1990).

It should also be mentioned that, when the material is *elastically isotropic*, Σ is symmetric (Mandel, 1971b; Hahn, 1974). A simple proof of this is given in the following. Using the polar decomposition theorem we can write $\mathbf{F}^e = \mathbf{V}^e\cdot\mathbf{R}^e$, where $\mathbf{V}^e$ is the symmetric left elastic stretch tensor. The equation for Σ now becomes

$$\Sigma = \mathbf{R}^{e\mathrm{T}}\cdot\mathbf{V}^{e-1}\cdot\tau\cdot\mathbf{V}^e\cdot\mathbf{R}^e. \tag{13}$$

In isotropic elasticity, the principal directions of τ coincide with those of $\mathbf{V}^e$ (coaxiality), so that $\tau\cdot\mathbf{V}^e = \mathbf{V}^e\cdot\tau$ and (13) reduces to

$$\Sigma = \mathbf{R}^{e\mathrm{T}}\cdot\tau\cdot\mathbf{R}^e. \tag{14}$$

In that case Σ is the symmetric and elastic work conjugate to the Lagrangian logarithmic strain measure (Hill, 1968; Freund, 1970). Similarly, (11) reduces to

$$\Sigma' = \mathbf{R}^{e\mathrm{T}}\cdot\tau'\cdot\mathbf{R}^e, \tag{15}$$

and Σ' becomes independent of p. It should be emphasized, however, that in *elastically anisotropic* materials Σ is not a symmetric tensor, neither is the right-hand side of (14) an elastic work conjugate to the Lagrangian logarithmic strain measure.

4. CONSTITUTIVE EQUATIONS

4.1. *Elastic equations*

We assume that an elastic potential Φ exists in the isoclinic configuration, so that

$$\mathbf{S}^e = \rho_i\frac{\partial\Phi}{\partial\mathbf{E}^e} \quad\text{or equivalently}\quad \bar{\mathbf{S}}^e = \rho_0\frac{\partial\Phi}{\partial\mathbf{E}^e}, \tag{16}$$

where $\mathbf{S}^e = J^e\mathbf{F}^{e-1}\cdot\sigma\cdot\mathbf{F}^{e-\mathrm{T}}$ is the elastic second Piola–Kirchhoff defined at $\mathfrak{B}_i$, $\bar{\mathbf{S}}^e = (J/J^e)\mathbf{S}^e = \mathbf{F}^{e-1}\cdot\tau\cdot\mathbf{F}^{e-\mathrm{T}}$, and ρ_i is the mass density in the isoclinic configuration $\mathfrak{B}_i$.

We consider a transversely isotropic material and denote by $\mathbf{m}$ the unit vector along the local direction of axial symmetry in the isoclinic configuration $\mathfrak{B}_i$. For such a material, the elastic potential Φ must be a function of the invariants $\mathrm{tr}(\mathbf{E}^e)$, $\mathrm{tr}(\mathbf{E}^{e2})$, $\mathrm{tr}(\mathbf{E}^{e3})$, $\mathrm{tr}(\mathbf{E}^e\cdot\mathbf{A})$ and $\mathrm{tr}(\mathbf{E}^{e2}\cdot\mathbf{A})$, where $\mathbf{A}$ is the orientation tensor $\mathbf{A} = \mathbf{mm}$ [see, for example, Spencer (1971)].

In the following, we derive the rate form of the above hyperelastic constitutive equation. Differentiating (16) with respect to time and taking into account that, at each material point, **A** remains fixed in the isoclinic configuration, we find

$$\dot{\bar{\mathbf{S}}}^e = \mathbf{C} : \dot{\mathbf{E}}^e, \quad \text{where} \quad \mathbf{C} = \rho_0 \frac{\partial^2 \Phi}{\partial \mathbf{E}^e \, \partial \mathbf{E}^e}. \tag{17}$$

After some lengthy but otherwise straightforward calculations, we find that (17) can be written in the current configuration $\mathfrak{B}$ as (Dafalias, 1985b; Needleman, 1985):

$$\overset{\nabla}{\boldsymbol{\tau}} = \mathfrak{L}^e_\tau : \mathbf{D}^e, \tag{18}$$

where

$$\overset{\nabla}{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} + \boldsymbol{\tau} \cdot \mathbf{W}^* - \mathbf{W}^* \cdot \boldsymbol{\tau}, \qquad \mathfrak{L}^e_{ijkl} = F^e_{im} F^e_{jn} F^e_{kp} F^e_{lq} C_{mnpq} \tag{19, 20}$$

$$\mathfrak{L}^e_\tau = \mathfrak{L}^e + \mathbf{T}, \tag{21}$$

and

$$T_{ijkl} = \tfrac{1}{2}(\tau_{ik}\delta_{jl} + \tau_{il}\delta_{jk} + \delta_{ik}\tau_{jl} + \delta_{il}\tau_{jk}). \tag{22}$$

When $\bar{\mathbf{S}}^e$ is a linear function of $\mathbf{E}^e$, Φ should be a quadratic function of the components of $\mathbf{E}^e$, and, therefore, it must be of the form (Spencer, 1972, 1984):

$$\rho_0 \Phi = a \operatorname{tr}^2(\mathbf{E}^e) + b \operatorname{tr}(\mathbf{E}^{e2}) + c \operatorname{tr}^2(\mathbf{E}^e \cdot \mathbf{A}) + d \operatorname{tr}(\mathbf{E}^{e2} \cdot A) + e \operatorname{tr}(\mathbf{E}^e) \operatorname{tr}(\mathbf{E}^e \cdot \mathbf{A}), \tag{23}$$

where a, b, c, d and e are the five independent elastic constants of the transversely isotropic material. In that case

$$\bar{\mathbf{S}}^e = \mathbf{C} : \mathbf{E}^e, \tag{24}$$

and

$$\mathbf{C} = 2a\mathbf{II} + 2b\mathbf{J} + 2c\mathbf{AA} + d\mathbf{P} + e(\mathbf{IA} + \mathbf{AI}), \tag{25}$$

where **J** is the symmetric unit fourth order tensor as defined in the Introduction, and

$$P_{ijkl} = \tfrac{1}{2}(A_{ik}\delta_{jl} + A_{il}\delta_{jk} + \delta_{ik}A_{jl} + \delta_{il}A_{jk}). \tag{26}$$

The constants a, b, c, d and e can be written in terms of the "standard" elastic constants E_{11}, μ_{12}, μ_{23}, K_{23} and ν_{12} as defined, for example, in Christensen's (1979) book, where the x_1-axis is in the direction of transverse isotropy; it can be readily shown that

$$a = \tfrac{1}{2}(K_{23} - \mu_{23}), \qquad b = \mu_{23}, \qquad c = \tfrac{1}{2}E_{11} + \tfrac{1}{2}\mu_{23} - 2\mu_{12} + \tfrac{1}{2}(1 - 2\nu_{12})^2 K_{23}, \tag{27, 28, 29}$$

$$d = 2(\mu_{12} - \mu_{23}), \qquad e = \mu_{23} - (1 - 2\nu_{12})K_{23}. \tag{30, 31}$$

The general form of elastic constitutive equations for anisotropic materials is discussed in detail in the work of Boehler (1975, 1979) and Walpole (1981).

4.2. *Plastic equations*

The yield condition and the flow rule are written in the isoclinic configuration in terms of the plastic work conjugate quantities Σ and $\mathbf{L}^p_i = \mathbf{D}^p_i + \mathbf{W}^p_i$. One of the advantages of working in terms of work-conjugate quantities is that a "normality rule" in conjugate stress and strain variables applies either for every choice of reference state and strain measure or for none (Hill and Rice, 1973). It should be emphasized, however, that Hill and Rice (1973) use a *fixed* reference configuration and express their normality rule and the elastic–plastic decomposition of their strain rate in terms of work-conjugate variables and the associated tensor of the elastic compliance, whereas, here, $\mathbf{\Sigma}$, $\mathbf{L}_i^p$, and the strain-energy function are all defined in the isoclinic configuration $\mathfrak{B}_i$, which is *evolving* with $\mathbf{F}^p$.

The formulation presented in the following makes use of the representation theorems for isotropic functions developed by Wang (1970a,b), Smith (1971) and Boehler (1977), and parallels the works of Hahn (1974), Loret (1983) and Dafalias (1985a). Anticipating the constitutive equations for the plastic spin developed in the following section, we choose to write explicitly the representations of the plastic part of the deformation rate and the plastic spin (instead of simply referring to the aforementioned works) in order to derive several simple relationships among the coefficients of the constitutive equations for $\mathbf{D}_i^p$ and $\mathbf{W}_i^p$ (eqns (37) and (38) below).

We write the yield condition in the isoclinic configuration $\mathfrak{B}_i$ as

$$\hat{f}(\mathbf{\Sigma}, \mathbf{A}, s) = f(\mathbf{\Sigma}_s, \mathbf{\Sigma}_a, \mathbf{A}, s) = 0, \tag{32}$$

where $\mathbf{\Sigma}_s$ and $\mathbf{\Sigma}_a$ are the symmetric and anti-symmetric parts of $\mathbf{\Sigma}$, s denotes a *collection* of state variables (which can be scalars or tensors of any order), and $\hat{f}$ and f are isotropic functions (Liu, 1982).

The constitutive equations for $\mathbf{D}_i^p$ and $\mathbf{W}_i^p$, and the evolution equations for the hardening parameters s are written as

$$\mathbf{D}_i^p = \langle \dot{\lambda} \rangle \mathbf{N}_i^p(\mathbf{\Sigma}, \mathbf{A}, s), \quad \mathbf{W}_i^p = \langle \dot{\lambda} \rangle \mathbf{\Omega}_i^p(\mathbf{\Sigma}, \mathbf{A}, s) \quad \text{and} \quad \dot{s} = \langle \dot{\lambda} \rangle \bar{s}(\mathbf{\Sigma}, \mathbf{A}, s), \tag{33}$$

where $\mathbf{N}_i^p$, $\mathbf{\Omega}_i^p$ and $\bar{s}$ are isotropic functions of their arguments, $\dot{\lambda}$ is a loading parameter, $\langle \dot{\lambda} \rangle = \dot{\lambda}$ if $\dot{\lambda} > 0$ and $\langle \dot{\lambda} \rangle = 0$ if $\dot{\lambda} \leqslant 0$.

We mention again that the material time derivatives used in the evolution equations of the state variables in (33_3) can also be viewed as co-rotational with the substructure, since the orientation of the latter is fixed at all times in the isoclinic configuration.

The corresponding equations in the current configuration would be

$$\mathbf{D}^p = \langle \dot{\lambda} \rangle \mathbf{N}^p \quad \text{and} \quad \mathbf{W}^p = \langle \dot{\lambda} \rangle \mathbf{\Omega}^p, \tag{34}$$

where

$$\mathbf{N}^p = [\mathbf{F}^e \cdot (\mathbf{N}_i^p + \mathbf{\Omega}_i^p) \cdot \mathbf{F}^{e-1}]_s \quad \text{and} \quad \mathbf{\Omega}^p = [\mathbf{F}^e \cdot (\mathbf{N}_i^p + \mathbf{\Omega}_i^p) \cdot \mathbf{F}^{e-1}]_a. \tag{35}$$

If we now assume that all state variables s are *scalar* and use the representation theorems for isotropic functions, we conclude that, for a transversely isotropic material, the most general form of f, $\mathbf{D}_i^p$ and $\mathbf{W}_i^p$ is

$$\begin{aligned} f(&\mathrm{tr}\,(\mathbf{\Sigma}_s), \mathrm{tr}\,(\mathbf{\Sigma}_s^2), \mathrm{tr}\,(\mathbf{\Sigma}_s^3), \mathrm{tr}\,(\mathbf{\Sigma}_a^2), \mathrm{tr}\,(\mathbf{\Sigma}_s \cdot \mathbf{A}), \mathrm{tr}\,(\mathbf{\Sigma}_s^2 \cdot \mathbf{A}), \mathrm{tr}\,(\mathbf{\Sigma}_s \cdot \mathbf{\Sigma}_a^2), \\ &\mathrm{tr}\,(\mathbf{\Sigma}_s^2 \cdot \mathbf{\Sigma}_a^2), \mathrm{tr}\,(\mathbf{A} \cdot \mathbf{\Sigma}_a^2), \mathrm{tr}\,(\mathbf{\Sigma}_s \cdot \mathbf{A} \cdot \mathbf{\Sigma}_a), \mathrm{tr}\,(\mathbf{\Sigma}_s^2 \cdot \mathbf{A} \cdot \mathbf{\Sigma}_a), \\ &\mathrm{tr}\,(\mathbf{\Sigma}_s \cdot \mathbf{\Sigma}_a^2 \cdot \mathbf{A} \cdot \mathbf{\Sigma}_a^2), s) = 0, \end{aligned} \tag{36}$$

$$\begin{aligned} \mathbf{D}_i^p = {} & \phi_1 \mathbf{I} + \phi_2 \mathbf{\Sigma}_s + \phi_3 \mathbf{\Sigma}_s^2 + \phi_4 \mathbf{A} + \phi_5 \mathbf{\Sigma}_a^2 + \phi_6 (\mathbf{\Sigma}_s \cdot \mathbf{A} + \mathbf{A} \cdot \mathbf{\Sigma}_s) \\ & + \phi_7 (\mathbf{\Sigma}_s^2 \cdot \mathbf{A} + \mathbf{A} \cdot \mathbf{\Sigma}_s^2) + \phi_8 (\mathbf{\Sigma}_s \cdot \mathbf{\Sigma}_a - \mathbf{\Sigma}_a \cdot \mathbf{\Sigma}_s) + \phi_9 \mathbf{\Sigma}_a \cdot \mathbf{\Sigma}_s \cdot \mathbf{\Sigma}_a \\ & + \phi_{10} (\mathbf{\Sigma}_s^2 \cdot \mathbf{\Sigma}_a - \mathbf{\Sigma}_a \cdot \mathbf{\Sigma}_s^2) + \phi_{11} (\mathbf{\Sigma}_a \cdot \mathbf{\Sigma}_s \cdot \mathbf{\Sigma}_a^2 - \mathbf{\Sigma}_a^2 \cdot \mathbf{\Sigma}_s \cdot \mathbf{\Sigma}_a) \\ & + \phi_{12} (\mathbf{A} \cdot \mathbf{\Sigma}_a - \mathbf{\Sigma}_a \cdot \mathbf{A}) + \phi_{13} \mathbf{\Sigma}_a \cdot \mathbf{A} \cdot \mathbf{\Sigma}_a + \phi_{14} (\mathbf{\Sigma}_a \cdot \mathbf{A} \cdot \mathbf{\Sigma}_a^2 - \mathbf{\Sigma}_a^2 \cdot \mathbf{A}_s \cdot \mathbf{\Sigma}_a), \end{aligned} \tag{37}$$

and

$$
\begin{aligned}
\mathbf{W}_i^p = {} & \eta_1 \boldsymbol{\Sigma}_a + \eta_2(\boldsymbol{\Sigma}_s \cdot \mathbf{A} - \mathbf{A} \cdot \boldsymbol{\Sigma}_s) + \eta_3(\boldsymbol{\Sigma}_s^2 \cdot \mathbf{A} - \mathbf{A} \cdot \boldsymbol{\Sigma}_s^2) \\
& + \eta_4(\boldsymbol{\Sigma}_s \cdot \mathbf{A} \cdot \boldsymbol{\Sigma}_s^2 - \boldsymbol{\Sigma}_s^2 \cdot \mathbf{A} \cdot \boldsymbol{\Sigma}_s) + \eta_5(\boldsymbol{\Sigma}_s \cdot \boldsymbol{\Sigma}_a + \boldsymbol{\Sigma}_a \cdot \boldsymbol{\Sigma}_s) \\
& + \eta_6(\boldsymbol{\Sigma}_s \cdot \boldsymbol{\Sigma}_a^2 - \boldsymbol{\Sigma}_a^2 \cdot \boldsymbol{\Sigma}_s) + \eta_7(\boldsymbol{\Sigma}_a \cdot \mathbf{A} + \mathbf{A} \cdot \boldsymbol{\Sigma}_a) \\
& + \eta_8(\mathbf{A} \cdot \boldsymbol{\Sigma}_a^2 - \boldsymbol{\Sigma}_a^2 \cdot \mathbf{A}),
\end{aligned}
\tag{38}
$$

where the ϕ_is and the η_is are scalar-valued functions of the invariants that enter the argument of f in (36).

In the formulation above, the non-symmetric stress tensor $\boldsymbol{\Sigma}$ is decomposed into its symmetric and anti-symmetric parts so that the standard form of the representation theorems can be used. Of course, the general expressions (36)–(38) can be written in terms of $\boldsymbol{\Sigma}$ alone if one substitutes $\boldsymbol{\Sigma}_s = (\boldsymbol{\Sigma}+\boldsymbol{\Sigma}^T)/2$ and $\boldsymbol{\Sigma}_a = (\boldsymbol{\Sigma}-\boldsymbol{\Sigma}^T)/2$. It should also be noted that, when the elastic strains are small, $\boldsymbol{\Sigma}$ is symmetric to within terms of order elastic strain times stress (see Section 4.6 below).

Using the consistency condition $\dot{f} = 0$ and taking into account that, at each material point, $\mathbf{A}$ remains fixed in the isoclinic configuration, we find that (Aravas, 1991)

$$
\dot{\lambda} = \mathbf{r} : \mathbf{D}, \tag{39}
$$

where

$$
\mathbf{r} = \frac{\mathbf{N}^p : (\mathfrak{L}_i^e + \hat{\mathbf{T}})}{\mathfrak{H} + \mathbf{N}^p : (\mathfrak{L}_i^e + \hat{\mathbf{T}}) : \mathbf{N}^p}, \quad \mathbf{N}_i^p = \frac{\partial \hat{f}}{\partial \boldsymbol{\Sigma}}, \quad \mathfrak{H} = -\frac{\partial \hat{f}}{\partial s} \bar{s}, \tag{40}
$$

and

$$
\mathbf{N}^p = \mathbf{F}^{e\,-T} \cdot \mathbf{N}_i^p \cdot \mathbf{F}^{eT}, \quad \hat{T}_{ijkl} = \tau_{ik}\delta_{jl} - \tau_{il}\delta_{jk}. \tag{41}
$$

Summarizing, we write (34) as

$$
\mathbf{D}^p = h(\dot{\lambda})\mathbf{N}^p \mathbf{r} : \mathbf{D} \quad \text{and} \quad \mathbf{W}^p = h(\dot{\lambda})\boldsymbol{\Omega}^p \mathbf{r} : \mathbf{D}, \tag{42}
$$

where $h(\dot{\lambda})$ is the Heaviside step function.

We conclude this section by mentioning that in the works of Mandel (1971a), Sidoroff and Teodosiu (1986) and Van Der Giessen (1989a,b) the yield condition, which is written in terms of the non-symmetric stress $\boldsymbol{\Sigma}$ at $\mathfrak{B}_i$, is also used as a plastic potential for $\mathbf{L}_i^p$. The completeness of such a flow rule has been questioned by Lubliner (1986, 1990), in view of the fact that $\boldsymbol{\Sigma}$ is limited to a six-dimensional manifold. Furthermore, Lubliner (1986) has shown that the normality rule resulting from "the principle of maximum plastic dissipation" determines only the projection of $\mathbf{L}_i^p$ onto a six-dimensional subspace of the space of second order tensors, so that a constitutive equation for the plastic spin can be written independently.

4.3. *A constitutive equation for the plastic spin*

We consider a unit vector $\mathbf{m}_0$ attached to a material fiber in the direction of axial symmetry in the undeformed configuration $\mathfrak{B}_0$. The corresponding unit vector in the intermediate configuration $\mathfrak{B}_i$ is $\mathbf{m} = \mathbf{F}^p \cdot \mathbf{m}_0 / |\mathbf{F}^p \cdot \mathbf{m}_0|$. Since the intermediate configuration is isoclinic, we have shown that $\mathbf{m} = \mathbf{m}_0$, which actually means that $\mathbf{m}_0$ (or $\mathbf{m}$) is an eigenvector of $\mathbf{F}^p$. Furthermore, the rate of change of $\mathbf{m}$ in the isoclinic configuration vanishes, so that

$$
\dot{\mathbf{m}} = (\mathbf{W}_i^p + \mathbf{D}_i^p \cdot \mathbf{m}\mathbf{m} - \mathbf{m}\mathbf{m} \cdot \mathbf{D}_i^p) \cdot \mathbf{m} = \mathbf{0}. \tag{43}
$$

The above equation shows that $\mathbf{m}$ is the axial vector of the anti-symmetric tensor $\mathbf{W}_i^p +$

$\mathbf{D}_i^p \cdot \mathbf{mm} - \mathbf{mm} \cdot \mathbf{D}_i^p$, and, therefore, we have the representation [see, for example, Ogden (1984)]

$$\mathbf{W}_i^p + \mathbf{D}_i^p \cdot \mathbf{mm} - \mathbf{mm} \cdot \mathbf{D}_i^p = \alpha(\mathbf{m}_2\mathbf{m}_3 - \mathbf{m}_3\mathbf{m}_2), \tag{44}$$

where $\mathbf{m}_2$, $\mathbf{m}_3$ and $\mathbf{m}$ form an orthonormal basis, and α is an arbitrary constant. The right-hand side of (44) is a spin about $\mathbf{m}$ at $\mathfrak{B}_i$ and reflects the arbitrary rigid rotation about the local axis of symmetry that can be superposed on the isoclinic configuration as discussed in Section 2. Transverse isotropy implies rotational symmetry about $\mathbf{m}$, so that a spin about $\mathbf{m}$ is inconsequential; therefore, α can be set to zero in (44), which reduces to

$$\mathbf{W}_i^p = \mathbf{A} \cdot \mathbf{D}_i^p - \mathbf{D}_i^p \cdot \mathbf{A}. \tag{45}$$

Aravas and Aifantis (1991) have also derived (45) for the special case of plane motions. In view of (45), one might be tempted to reach the conclusion that a constitutive equation for $\mathbf{W}_i^p$ (or $\mathbf{W}^p$) is not necessary, since $\mathbf{W}_i^p$ can be determined in terms of $\mathbf{D}_i^p$. We emphasize, however, that (45) must be viewed as a constitutive equation for $\mathbf{W}_i^p$, since it is based on the fact that the direction of transverse isotropy at each material point follows the deformation of the continuum, and this is a constitutive assumption by itself.

Equation (45) shows that the scalar functions in (37) and (38) must be such that

$$\eta_2 = -(\phi_2+\phi_6), \quad \eta_3 = -(\phi_3+\phi_7), \quad \eta_7 = \phi_{12}, \quad \eta_8 = \phi_5, \tag{46}$$

and

$$\eta_1 = \eta_4 = \eta_5 = \eta_6 = \phi_8 = \phi_9 = \phi_{10} = \phi_{11} = \phi_{14} = 0. \tag{47}$$

The most general form of $\mathbf{D}_i^p$ and $\mathbf{W}_i^p$ now becomes

$$\begin{aligned}\mathbf{D}_i^p = {} & \phi_1\mathbf{I} + \phi_2\mathbf{\Sigma}_s + \phi_3\mathbf{\Sigma}_s^2 + \phi_4\mathbf{A} + \phi_5\mathbf{\Sigma}_a^2 + \phi_6(\mathbf{\Sigma}_s \cdot \mathbf{A} + \mathbf{A} \cdot \mathbf{\Sigma}_s) \\ & + \phi_7(\mathbf{\Sigma}_s^2 \cdot \mathbf{A} + \mathbf{A} \cdot \mathbf{\Sigma}_s^2) + \phi_{12}(\mathbf{A} \cdot \mathbf{\Sigma}_a - \mathbf{\Sigma}_a \cdot \mathbf{A}) + \phi_{13}\mathbf{\Sigma}_a \cdot \mathbf{A} \cdot \mathbf{\Sigma}_a,\end{aligned} \tag{48}$$

and

$$\begin{aligned}\mathbf{W}_i^p = {} & -(\phi_2+\phi_6)(\mathbf{\Sigma}_s \cdot \mathbf{A} - \mathbf{A} \cdot \mathbf{\Sigma}_s) - (\phi_3+\phi_7)(\mathbf{\Sigma}_s^2 \cdot \mathbf{A} - \mathbf{A} \cdot \mathbf{\Sigma}_s^2) \\ & + \phi_{12}(\mathbf{\Sigma}_a \cdot \mathbf{A} + \mathbf{A} \cdot \mathbf{\Sigma}_a) + \phi_5(\mathbf{A} \cdot \mathbf{\Sigma}_a^2 - \mathbf{\Sigma}_a^2 \cdot \mathbf{A}).\end{aligned} \tag{49}$$

4.4. *Rate form of the elastoplastic equations*

Using the rate form of the elasticity equation (18) together with the additive strain rate decomposition (3) and the expression (42_1) for $\mathbf{D}^p$ we find

$$\overset{\circ}{\boldsymbol{\tau}} = \mathfrak{L}_\tau^e : \mathbf{D}^e = \mathfrak{L}_\tau^e : (\mathbf{D} - \mathbf{D}^p) = \mathfrak{L}_\tau^e : (\mathbf{D} - h(\dot{\lambda})\mathbf{N}^p\mathbf{r} : \mathbf{D}), \tag{50}$$

or

$$\overset{\circ}{\boldsymbol{\tau}} = \mathfrak{L} : \mathbf{D}, \tag{51}$$

where

$$\mathfrak{L} = \mathfrak{L}_\tau^e : (\mathbf{J} - h(\dot{\lambda})\mathbf{N}^p\mathbf{r}). \tag{52}$$

Equation (51) can also be written as

$$\overset{\nabla}{\boldsymbol{\tau}} = \mathfrak{L}_J : \mathbf{D}, \tag{53}$$

where a superposed ∇ denotes the Jaumann derivative, and

$$\mathfrak{L}_J = \mathfrak{L} + h(\dot{\lambda})(\boldsymbol{\tau}\cdot\boldsymbol{\Omega}^p - \boldsymbol{\Omega}^p\cdot\boldsymbol{\tau})\mathbf{r}. \tag{54}$$

4.5. *Hill's anisotropic yield criterion*

A special case of the yield condition (32) is that due to Hill (1948, 1950), which is written in the following in terms of the symmetric part of $\boldsymbol{\Sigma}$. When the x_1-axis is along the axis of transverse isotropy, Hill's criterion is written in the isoclinic configuration as

$$F(\Sigma_{s22}-\Sigma_{s33})^2 + G[(\Sigma_{s11}-\Sigma_{s22})^2 + (\Sigma_{s11}-\Sigma_{s33})^2] + 2(G+2F)\Sigma_{s23}^2 + 2M(\Sigma_{s12}^2+\Sigma_{s13}^2) - \sigma_y^2(\bar{\varepsilon}^p) = 0, \tag{55}$$

where F, G and M are constants, σ_y is a material parameter characteristic of the current state of hardening, and $\bar{\varepsilon}^p$ is the equivalent plastic strain to be defined in the following. The tensile yield stresses in the axial and transverse directions are $\sigma_{y1} = \sigma_y/\sqrt{2G}$ and $\sigma_{yt} = \sigma_y/\sqrt{F+G}$ respectively; the corresponding yield stresses in shear are $\tau_{y1} = \sigma_y/\sqrt{2M}$ and $\tau_{yt} = \sigma_y/\sqrt{2(2F+G)}$.

Dafalias (1987b) and Dafalias and Rashid (1989) have shown that the above equation is indeed of the form (36) and that, with respect to an arbitrary Cartesian coordinate system, can be written as

$$(G+2F)\,\mathrm{tr}\,(\boldsymbol{\Sigma}_s'^2) + 2(M-G-2F)\,\mathrm{tr}\,(\boldsymbol{\Sigma}_s'^2\cdot\mathbf{A}) + (5G+F-2M)\,\mathrm{tr}^2\,(\boldsymbol{\Sigma}_s'\cdot\mathbf{A}) - \sigma_y^2(\bar{\varepsilon}^p) = 0. \tag{56}$$

The yield condition (56) reduces to that of von Mises when $F = G = 1/2$ and $M = 3/2$. The above equation can also be used as a plastic potential in the isoclinic configuration, in which case

$$\mathbf{D}_i^p = \langle\dot{\lambda}\rangle\frac{\partial f}{\partial \boldsymbol{\Sigma}_s} = \phi_1\mathbf{I} + \phi_2\boldsymbol{\Sigma}_s' + \phi_4\mathbf{A}' + \phi_6(\boldsymbol{\Sigma}_s'\cdot\mathbf{A} + \mathbf{A}\cdot\boldsymbol{\Sigma}_s'), \tag{57}$$

where

$$\phi_1 = -\tfrac{4}{3}\langle\dot{\lambda}\rangle(M-G-2F)\,\mathrm{tr}\,(\boldsymbol{\Sigma}_s'\cdot\mathbf{A}), \qquad \phi_2 = 2\langle\dot{\lambda}\rangle(G+2F), \tag{58, 59}$$

$$\phi_4 = 2\langle\dot{\lambda}\rangle(5G+F-2M)\,\mathrm{tr}\,(\boldsymbol{\Sigma}_s'\cdot\mathbf{A}), \qquad \phi_6 = 2\langle\dot{\lambda}\rangle(M-G-2F). \tag{60, 61}$$

Using equation (45) we find that the corresponding equation for the plastic spin is

$$\mathbf{W}_i^p = 2\langle\dot{\lambda}\rangle M(\mathbf{A}\cdot\boldsymbol{\Sigma}_s - \boldsymbol{\Sigma}_s\cdot\mathbf{A}). \tag{62}$$

It should be noted that the above equation is consistent with Dafalias' (1983, 1984) general formulation for transversely isotropic materials; in fact, when Dafalias' plastic spin parameter η is set to $2M$, eqn (62), above, is recovered.

Following Hill (1950) we define the equivalent stress $\bar{\sigma}$ in the isoclinic configuration as

$$\bar{\sigma}^2 = \frac{3}{2(F+2G)}[(G+2F)\,\mathrm{tr}\,(\boldsymbol{\Sigma}_s'^2) + 2(M-G-2F)\,\mathrm{tr}\,(\boldsymbol{\Sigma}_s'^2\cdot\mathbf{A}) + (5G+F-2M)\,\mathrm{tr}^2\,(\boldsymbol{\Sigma}_s'^2\cdot\mathbf{A})]. \tag{63}$$

It can be readily shown that the corresponding plastic work conjugate equivalent plastic strain rate $\dot{\bar{\varepsilon}}^{p}$ is

$$\dot{\bar{\varepsilon}}^{p} = \sqrt{\frac{2}{3}\frac{F+2G}{G+2F}\left[\operatorname{tr}(\mathbf{D}_{i}^{p2})+2\left(\frac{G+2F}{M}-1\right)\operatorname{tr}(\mathbf{D}_{i}^{p2}\cdot\mathbf{A}) + \left(\frac{F+G}{G}-2\frac{G+2F}{M}\right)\operatorname{tr}^{2}(\mathbf{D}_{i}^{p}\cdot\mathbf{A})\right]^{1/2}}, \quad (64)$$

so that

$$\bar{\sigma}\dot{\bar{\varepsilon}}^{p} = \boldsymbol{\Sigma}_{s} : \mathbf{D}_{i}^{p}. \quad (65)$$

4.6. *Small elastic strains*

In this section we consider the case of infinitesimal elastic strains and show that the formulation simplifies considerably due to the secondary role of elasticity. In this case we have ($\mathbf{F}^{e} = \mathbf{R}^{e}\cdot\mathbf{U}^{e}$)

$$\mathbf{U}^{e} = \mathbf{I}+\varepsilon\hat{\mathbf{U}}^{e}, \quad (66)$$

where

$$|\varepsilon| \ll 1, \quad \hat{\mathbf{U}}^{eT} = \hat{\mathbf{U}}^{e} \quad \text{and} \quad \sup \|\hat{\mathbf{U}}^{e}\| = O(1). \quad (67)$$

The kinematics can now be written as

$$\mathbf{F}^{e} = \mathbf{R}^{e}+\varepsilon\mathbf{R}^{e}\cdot\hat{\mathbf{U}}^{e}, \qquad J^{e} = 1+O(\varepsilon), \qquad \mathbf{D}^{e} = \varepsilon\mathbf{R}^{e}\cdot\dot{\hat{\mathbf{U}}}^{e}\cdot\mathbf{R}^{eT}+O(\varepsilon^{2}\dot{\mathbf{R}}^{e},\varepsilon^{2}\dot{\hat{\mathbf{U}}}^{e}), \quad (68, 69, 70)$$

$$\mathbf{W}^{e} = O(\varepsilon^{2}\dot{\mathbf{R}}^{e},\varepsilon^{2}\dot{\hat{\mathbf{U}}}^{e}), \qquad \mathbf{W}^{*} = \dot{\mathbf{R}}^{e}\cdot\mathbf{R}^{eT}+O(\varepsilon^{2}\dot{\mathbf{R}}^{e},\varepsilon^{2}\dot{\hat{\mathbf{U}}}^{e}), \quad (71, 72)$$

$$\mathbf{D}^{p} = \mathbf{R}^{e}\cdot\mathbf{D}_{i}^{p}\cdot\mathbf{R}^{eT}+O(\varepsilon\mathbf{W}_{i}^{p}), \qquad \mathbf{W}^{p} = \mathbf{R}^{e}\cdot\mathbf{W}_{i}^{p}\cdot\mathbf{R}^{eT}+O(\varepsilon\mathbf{D}_{i}^{p}). \quad (73, 74)$$

In most metals, the elastic strains are of the order of the yield strain, i.e. $\varepsilon = 10^{-3}$–10^{-2}. Also, all stress components are of the order of the yield stress or smaller, and the elastic moduli $\mathbf{C}$ are $O(\sigma/\varepsilon)$. The expressions for elastic moduli (30) and (31) can now be written as

$$\mathfrak{L}^{e} = \hat{\mathbf{C}}+O(\sigma), \qquad \mathfrak{L}_{t}^{e} = \hat{\mathbf{C}}+O(\sigma), \quad (75, 76)$$

where

$$\hat{C}_{ijkl} = R_{im}^{e}R_{jn}^{e}R_{kp}^{e}R_{lq}^{e}C_{mnpq}. \quad (77)$$

Furthermore,

$$\boldsymbol{\Sigma} = \bar{\mathbf{S}}^{e}+O(\varepsilon\sigma) = \mathbf{R}^{eT}\cdot\boldsymbol{\tau}\cdot\mathbf{R}^{e}+O(\varepsilon\sigma). \quad (78)$$

Equations (37) and (38) for $\mathbf{D}_{i}^{p}$ and $\mathbf{W}_{i}^{p}$ can now be written in the current configuration as

$$\mathbf{D}^{p} = \phi_{1}\mathbf{I}+\phi_{2}\boldsymbol{\tau}+\phi_{3}\boldsymbol{\tau}^{2}+\phi_{4}\mathbf{a}+\phi_{6}(\boldsymbol{\tau}\cdot\mathbf{a}+\mathbf{a}\cdot\boldsymbol{\tau})+\phi_{7}(\boldsymbol{\tau}^{2}\cdot\mathbf{a}+\mathbf{a}\cdot\boldsymbol{\tau}^{2})+O(\varepsilon\mathbf{L}_{i}^{p}), \quad (79)$$

and

$$\mathbf{W}^p = \eta_2(\boldsymbol{\tau}\cdot\mathbf{a}-\mathbf{a}\cdot\boldsymbol{\tau})+\eta_3(\boldsymbol{\tau}^2\cdot\mathbf{a}-\mathbf{a}\cdot\boldsymbol{\tau}^2)+\eta_4(\boldsymbol{\tau}\cdot\mathbf{a}\cdot\boldsymbol{\tau}^2-\boldsymbol{\tau}^2\cdot\mathbf{a}\cdot\boldsymbol{\tau})+O(\varepsilon\mathbf{L}_i^p), \tag{80}$$

where $\mathbf{a} = \mathbf{R}^e\cdot\mathbf{A}\cdot\mathbf{R}^{eT}$.

For definiteness, we assume now that the yield condition $\hat{f}(\boldsymbol{\Sigma},\mathbf{A},s) = 0$ has dimensions of stress; in the case of Hill's yield criterion, this amounts to assuming that σ_y has dimensions of stress and dividing (56) through by σ_y. Using (78) and taking into account that the yield function f is an isotropic function of its arguments, we can readily show that

$$\hat{f}(\boldsymbol{\Sigma},\mathbf{A},s) = \hat{f}(\bar{\mathbf{S}}^e,\mathbf{A},s)+O(\varepsilon\sigma) = \hat{f}(\boldsymbol{\tau},\mathbf{a},s)+O(\varepsilon\sigma) = 0. \tag{81}$$

When the yield function is normalized as mentioned above, we can also show that

$$\mathbf{N}^n = \mathbf{R}^e\cdot\mathbf{N}_i^n\cdot\mathbf{R}^{eT}+O(\varepsilon) = O(1). \tag{82}$$

Next, we assume that $\mathbf{N}_i^p = O(1)$ and, in view of (45) and (35) we also have that $\mathbf{N}^p = O(1)$ and $\boldsymbol{\Omega}^p = O(1)$. Equations (42), (52) and (54) can be written as

$$\mathbf{r} = \frac{\mathbf{N}^n:\hat{\mathbf{C}}+\mathbf{O}(\sigma)}{\mathfrak{H}+\mathbf{N}^n:\hat{\mathbf{C}}:\mathbf{N}^p+O(\sigma)}, \qquad \mathfrak{L} = \hat{\mathbf{C}}-h(\dot{\lambda})\frac{\hat{\mathbf{C}}:\mathbf{N}^p\mathbf{N}^n:\hat{\mathbf{C}}+O(\sigma^2/\varepsilon)}{\mathfrak{H}+\mathbf{N}^n:\hat{\mathbf{C}}:\mathbf{N}^p+O(\sigma)}+O(\sigma), \tag{83, 84}$$

and

$$\mathfrak{L}_J = \mathfrak{L}+O(\sigma). \tag{85}$$

Summarizing, we write eqn (53) as

$$\overset{\nabla}{\boldsymbol{\tau}} = \left(\hat{\mathbf{C}}-h(\dot{\lambda})\frac{\hat{\mathbf{C}}:\mathbf{N}^p\mathbf{N}^n:\hat{\mathbf{C}}+O(\sigma^2/\varepsilon)}{\mathfrak{H}+\mathbf{N}^n:\hat{\mathbf{C}}:\mathbf{N}^p+O(\sigma)}+O(\sigma)\right):\mathbf{D}. \tag{86}$$

A note of caution is relevant at this point. In the case of small elastic strains, it is common to write [see, for example, McMeeking and Rice (1975); Dafalias (1984)]

$$\overset{\nabla}{\boldsymbol{\tau}} = \left(\hat{\mathbf{C}}-h(\dot{\lambda})\frac{\hat{\mathbf{C}}:\mathbf{N}^p\mathbf{N}^n:\hat{\mathbf{C}}}{\mathfrak{H}+\mathbf{N}^n:\hat{\mathbf{C}}:\mathbf{N}^p}\right):\mathbf{D}, \tag{87}$$

where the above equation is consistent with (86). It should be noted, however, that in most metals $\mathfrak{H}$ is of the same order of magnitude as $\boldsymbol{\sigma}$; in such a case, when the $\mathfrak{H}$ term is retained in (87), the $O(\sigma)$ terms in the denominator of (86) cannot properly be disregarded. Therefore, when $\mathfrak{H} = O(\sigma)$ and (87) is used, the actual stress–strain relation in the hardening range is not precisely modelled. Furthermore, when the elastic strains are small and $\mathfrak{H} = O(\sigma)$, (87) with $\mathfrak{H} = 0$ (perfect plasticity) provides a consistent first order approximation of the exact constitutive equations of the hardening material. Rice (1970) reached the same conclusion for the case of small total (elastic and plastic) strains of a hypo-elastic–plastic material that obeys the Prandtl–Reuss equations.

5. FINITE ELEMENT FOUNDATION

Two different finite element formulations are discussed in the following.

In the *explicit* formulation of McMeeking and Rice (1975), the rate form of the equilibrium equations are enforced through the virtual power statement (Hill, 1959)

$$\int_V \frac{1}{J}[\overset{\nabla}{\boldsymbol{\tau}}:\mathbf{D}^*-\boldsymbol{\tau}:(2\mathbf{D}\cdot\mathbf{D}^*-\mathbf{L}^T\cdot\mathbf{L}^*)]\,\mathrm{d}V = \int_V \frac{1}{J}\dot{\mathbf{b}}_0\cdot\mathbf{v}^*\,\mathrm{d}V+\int_S \dot{\mathbf{T}}_0\cdot\mathbf{v}^*\frac{\mathrm{d}S_0}{\mathrm{d}S}\,\mathrm{d}S, \tag{88}$$

where V is the volume of the continuum and S its surface in the current configuration $\mathfrak{B}$,

S_0 is the surface in the undeformed configuration $\mathfrak{B}_0$, $\mathbf{T}_0 = \mathbf{N}_0 \cdot \mathbf{t}$, $\mathbf{N}_0$ is the unit normal to S_0, $\mathbf{t} = \mathbf{F}^{-1} \cdot \boldsymbol{\tau}$ is the nominal (first Piola–Kirchhoff) stress tensor, $\mathbf{b}_0$ is the body force per unit undeformed volume, $\mathbf{v}^*$ is an arbitrary virtual velocity field, and $\mathbf{D}^*$ and $\mathbf{L}^*$ are the corresponding deformation rate and velocity gradient respectively. Once the finite element discretization is introduced, the above equation reduces to a *linear* system of equations for the nodal displacement increments. In this case, the solution is developed incrementally, and the use of the so-called "equilibrium correction" at the end of each increment improves the accuracy of the numerical solution.

An alternative approach would be to start out with the weak form of the momentum balance, i.e. to use the principle of virtual work itself instead of its rate form (88) (Hibbitt, 1984; Moran *et al.*, 1990). In this case, the discretized equations consist of a system of *non-linear* equations which must be solved for the nodal displacements. A common solution technique is Newton's method, in which case the Jacobian plays the role of the incremental "stiffness matrix" in the calculations.

6. NUMERICAL INTEGRATION OF THE ELASTOPLASTIC EQUATIONS

In a finite element environment the solution of elastoplastic problems is developed incrementally and the constitutive equations are integrated at the element Gauss points. In a displacement-based finite element formulation the solution is deformation driven. At a material point, the solution $(\boldsymbol{\Sigma}_n, s_n, \mathbf{F}^p_n, \mathbf{F}_n)$ at time t_n as well as the deformation gradient $\mathbf{F}_{n+1}$ at time $t_{n+1} = t_n + \Delta t$ are supposed to be known and one has to determine the solution $(\boldsymbol{\Sigma}_{n+1}, s_{n+1}, \mathbf{F}^p_{n+1})$ at t_{n+1}.

In the following, we outline an algorithm for the numerical integration of the elasto-plastic equations, for the case where the yield function (32) depends on a set of state variables s, whereas $\mathbf{N}^p_f$, $\boldsymbol{\Omega}^p_f$ and $\bar{s}$ in (53) are independent of s; this is for example the case in Hill's model as formulated in Section 4.5, where $\bar{\varepsilon}^p$ is the only state variable. We start with eqn (7), which can be written as

$$\dot{\mathbf{F}}^p = \dot{\lambda}(\mathbf{N}^p_f + \boldsymbol{\Omega}^p_f) \cdot \mathbf{F}^p. \tag{89}$$

The direction of plastic flow $\mathbf{N}^p_f + \boldsymbol{\Omega}^p_f$ is assumed to be constant over the increment and equal to $(\mathbf{N}^p_f + \boldsymbol{\Omega}^p_f)_{n+1} \equiv \mathbf{B}_{n+1}$. Integration of the above equation then yields

$$\mathbf{F}^{p\,-1}_{n+1} = \mathbf{F}^{p\,-1}_n \cdot \exp(-\Delta\lambda \mathbf{B}_{n+1}) = \mathbf{F}^{p\,-1}_n \cdot [\mathbf{I} - \Delta\lambda \mathbf{B}_{n+1} + \tfrac{1}{2}\Delta\lambda^2 \mathbf{B}^2_{n+1} + O(\Delta\lambda^3)], \tag{90}$$

which is truncated to

$$\mathbf{F}^{p\,-1}_{n+1} = \mathbf{F}^{p\,-1}_n \cdot (\mathbf{I} - \Delta\lambda \mathbf{B}_{n+1} + \tfrac{1}{2}\Delta\lambda^2 \mathbf{B}^2_{n+1}). \tag{91}$$

The evolution equation for s (33_3) is integrated numerically using a backward Euler procedure and the constitutive calculations are based on an implicit treatment of the elastoplastic equations. A summary of the equations is given in the following:

$$\mathbf{F}^e_{n+1} = \mathbf{F}_{n+1} \cdot \mathbf{F}^{p\,-1}_{n+1} = \mathbf{F}^e_{\text{trial}} \cdot (\mathbf{I} - \Delta\lambda \mathbf{B}_{n+1} + \tfrac{1}{2}\Delta\lambda^2 \mathbf{B}^2_{n+1}), \tag{92}$$

$$\mathbf{C}^e_{n+1} = \mathbf{F}^{eT}_{n+1} \cdot \mathbf{F}^e_{n+1}, \qquad \mathbf{E}^e_{n+1} = \tfrac{1}{2}(\mathbf{C}^e_{n+1} - \mathbf{I}), \qquad \bar{\mathbf{S}}^e_{n+1} = \rho_0 \left(\frac{\partial \Phi}{\partial \mathbf{E}^e}\right)_{n+1}, \tag{93, 94, 95}$$

$$\boldsymbol{\Sigma}_{n+1} = \bar{\mathbf{S}}^e_{n+1} \cdot \mathbf{C}^e_{n+1}, \qquad s_{n+1} = s_n + \Delta\lambda \bar{s}(\boldsymbol{\Sigma}_{n+1}, \mathbf{A}), \qquad f(\boldsymbol{\Sigma}_{n+1}, \mathbf{A}, s_{n+1}) = 0, \tag{96, 97, 98}$$

where $\mathbf{F}^e_{\text{trial}} = \mathbf{F}_{n+1} \cdot \mathbf{F}^{p\,-1}_n$ is the trial elastic deformation gradient. We choose $\Delta\lambda$ as the primary unknown, treating the yield condition (98) as the basic equation for its determination. The solution is obtained using the secant method. Within the secant loop, for each value of $\Delta\lambda$ the corresponding Σ_{n+1} is determined from (92)–(96) using Newton's method.

Equation (91) implies that

$$\det(\mathbf{F}^{\mathrm{p}}_{n+1}) = [1+\Delta\lambda\,\mathrm{tr}(\mathbf{N}^{\mathrm{p}}_{l})_{n+1}+O(\Delta\lambda^2)]\det(\mathbf{F}^{\mathrm{p}}_{n}). \tag{99}$$

If the material is *plastically incompressible*, then $\mathrm{tr}(\mathbf{N}^{\mathrm{p}}_{l})_{n+1} = 0$, and the above equation becomes

$$\det(\mathbf{F}^{\mathrm{p}}_{n+1}) = [1+O(\Delta\lambda^2)]\det(\mathbf{F}^{\mathrm{p}}_{n}). \tag{100}$$

When the plastic equations are normalized as discussed in Section 4.6, λ is dimensionless and $\Delta\lambda$ is a measure of the "plastic strain increment" $\mathbf{D}^{\mathrm{p}}_{l}\Delta t$. Let $\Delta\lambda$ be $O(\delta)$ (δ small) at all increments; then we can show by induction that

$$\det(\mathbf{F}^{\mathrm{p}}_{n+1}) = 1+O(\delta^2). \tag{101}$$

With δ being of order 10^{-2}, the above equation shows that the integration algorithm preserves plastic incompressibility to within terms of order 10^{-4} in comparison to unity.

7. EXAMPLES

The finite element calculations are carried out using the ABAQUS general purpose finite element program (Hibbitt, 1984). This code provides a general interface so that the user may introduce his/her own constitutive model in a "user subroutine". The integration of the elastoplastic equations is carried out using the algorithm outlined in Section 6. The formulation is based on the weak form of the momentum balance and the discretized non-linear equations are solved using Newton's method. In our calculations, we approximate the Jacobian of the Newton scheme by the "tangent stiffness matrix" of McMeeking and Rice (1975). Such an approximation is first order accurate as the increment $\Delta\lambda \to 0$; it should be emphasized, however, that the aforementioned approximation influences the rate of convergence only and not the accuracy of the results.

7.1. *Plane strain simple shear*

In order to check the consistency of the finite element formulation and the numerical implementation of the algorithm described in the previous section we carry out a single element test subject to simple shear. We consider a plane strain simple shear transformation γ along the direction x_1 as shown in Fig. 3. The material is assumed to be rigid-perfectly–plastic obeying Hill's yield criterion (55) with associated flow rule and $\sigma_y = \sigma_0$ = constant. This problem has been analysed in detail by Dafalias and Rashid (1989); for completeness, the solution is outlined briefly in the following.

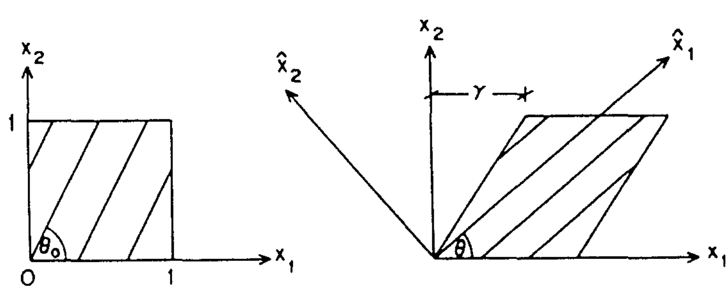

Fig. 3. Plane strain simple shear.

A coordinate system $\hat{x}_1 - \hat{x}_2$ is introduced in such a way that $\hat{x}_1$ is always in the direction of transverse isotropy in the deformed configuration. Referring to Fig. 3, we can readily show that

$$\tan\theta = \frac{1}{\gamma + \cot\theta_0}, \tag{102}$$

where θ is the angle of orientation of $\hat{x}_1$, and θ_0 the corresponding angle when $\gamma = 0$ (see Fig. 3). The only non-zero components of the rate of deformation **D** are

$$\hat{D}_{11} = -\hat{D}_{22} = \tfrac{1}{2}\dot{\gamma}\sin 2\theta \quad \text{and} \quad \hat{D}_{12} = \tfrac{1}{2}\dot{\gamma}\cos 2\theta, \tag{103}$$

where a circumflex indicates components with respect to the $\hat{x}_1 - \hat{x}_2$ coordinate system.

It should be noted that, because of incompressibility, the stresses can only be determined to within an arbitrary pressure. The yield condition and the flow rule can be written as

$$F(\hat{\sigma}'_{11} + 2\hat{\sigma}'_{22})^2 + G[(\hat{\sigma}'_{11} - \hat{\sigma}'_{22})^2 + (2\hat{\sigma}'_{11} + \hat{\sigma}'_{22})^2] + 2M\hat{\sigma}_{12}^2 - \sigma_0^2 = 0, \tag{104}$$

$$\tfrac{1}{2}\dot{\gamma}\sin 2\theta = 6\dot{\lambda}G\hat{\sigma}'_{11}, \qquad 0 = -2\dot{\lambda}[(F+2G)\hat{\sigma}'_{11} + (G+2F)\hat{\sigma}'_{22}], \tag{105, 106}$$

$$\tfrac{1}{2}\dot{\gamma}\cos 2\theta = 2\dot{\lambda}M\hat{\sigma}_{12}. \tag{107}$$

The solution to the above equations is readily found to be

$$\frac{\hat{\sigma}'_{11}}{c_{11}} = \frac{\hat{\sigma}'_{22}}{c_{22}} = \frac{\hat{\sigma}_{12}}{c_{12}} = \frac{\sigma_0}{[F(c_{11} + 2c_{22})^2 + G[(c_{11} - c_{22})^2 + (2c_{11} + c_{22})^2] + 2Mc_{12}^2]^{1/2}}, \tag{108}$$

where

$$c_{11} = \frac{\sin 2\theta}{12G}, \quad c_{22} = -\frac{F+2G}{G2F}c_{11} \quad \text{and} \quad c_{12} = \frac{\cos 2\theta}{4M}. \tag{109}$$

Finally, the components with respect to the x_1–x_2 coordinate system are found from

$$\sigma'_{11} = \tfrac{1}{2}(\hat{\sigma}'_{11} + \hat{\sigma}'_{22}) + \tfrac{1}{2}(\hat{\sigma}'_{11} - \hat{\sigma}'_{22})\cos 2\theta - \hat{\sigma}_{12}\sin 2\theta, \tag{110}$$

$$\sigma'_{22} = \tfrac{1}{2}(\hat{\sigma}'_{11} + \hat{\sigma}'_{22}) - \tfrac{1}{2}(\hat{\sigma}'_{11} - \hat{\sigma}'_{22})\cos 2\theta + \hat{\sigma}_{12}\sin 2\theta, \tag{111}$$

$$\sigma_{12} = \tfrac{1}{2}(\hat{\sigma}'_{11} - \hat{\sigma}'_{22})\sin 2\theta + \hat{\sigma}_{12}\cos 2\theta. \tag{112}$$

In Fig. 4 the deviatoric stress components are plotted versus γ for $F = 1.25$, $G = 0.5$, $M = 1.5$ and $\theta_0 = 90°$. The open symbols in that figure indicate the results of the elastic–plastic finite element calculations. The elastic moduli used in the finite element calculations are three orders of magnitude larger than σ_0 so that the role of elasticity becomes secondary. The results of the finite element calculations agree well with the exact solution (108).

7.2. *Plane strain extrusion*

We consider plane strain extrusion of a metal–matrix composite reinforced by short fibers which are initially aligned in the direction of extrusion. The height reduction is $\Delta h/h_0 = 0.25$, and the length of the reduction region is $L = 2h_0$ (see Fig. 5), where h_0 is the height at the start of the reduction region. The reduction area of the die is shaped in the form of a fifth order polynomial with zero slope and curvature at both ends. A rigid smooth piston pressing against the rear face of the billet providing the driving force. The coefficient of friction along the die-metal interface is assumed to vanish.

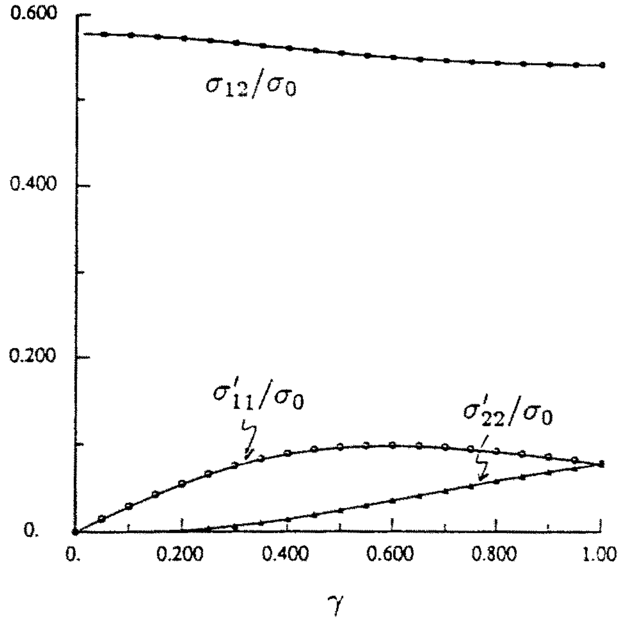

Fig. 4. Comparison of finite element and analytical solutions.

The material is assumed to be linear-elastic plastic obeying Hill's anisotropic yield criterion with associated flow rule as described in Section 4.5. The parameter σ_y is assumed to be a function of the equivalent plastic strain according to the relation

$$\frac{\sigma_y}{\sigma_0} = \left(1 + \frac{\bar{\varepsilon}^p}{\varepsilon_0}\right)^{1/n}, \tag{113}$$

where σ_0, ε_0 and n are material constants.

As a model material, we consider an aluminum matrix reinforced by short aligned boron fibers. Typical values of the elastic constants are $E = 70$ GPa, $\nu = 0.3$ for aluminum, and $E = 385$ GPa, $\nu = 0.2$ for boron, where E and ν and Young's modulus and Poisson's ratio respectively. Assuming that the fiber volume fraction is 20% and using the estimation

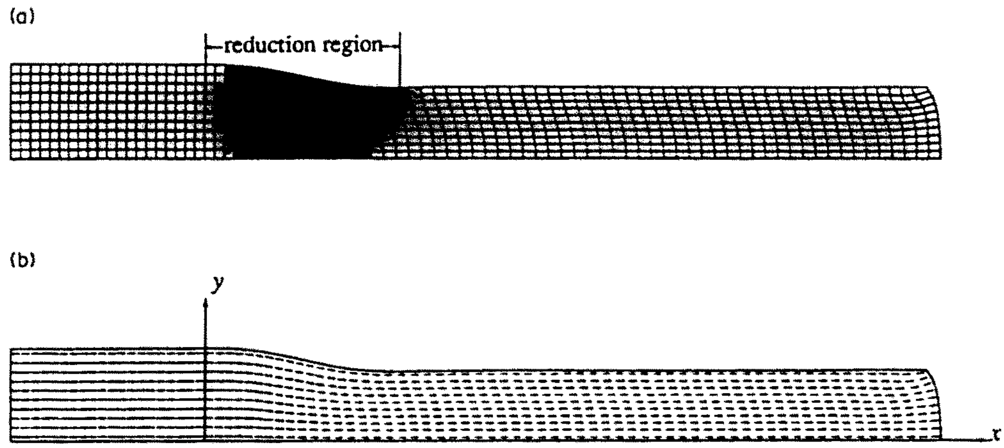

Fig. 5. (a) The deformed finite element mesh and the plastic zone. (b) Direction of transverse isotropy at the centroid of each element.

procedure described in Christensen (1979), we find the following approximate values for the elastic constants of the composite: $E_{11} = 135$ GPa, $\mu_{12} = 35$ GPa, $\mu_{23} = 30$ GPa, $K_{23} = 80$ GPa and $\nu_{12} = 0.27$. If we now assume that $\sigma_{yt}/\sigma_y \simeq 0.4$, $\tau_{y1}/\sigma_y \simeq 0.25$ and $\tau_{yt}/\sigma_y \simeq 0.4$, we find $F = 1.25$, $G = 0.5$ and $M = 8$. A typical value for σ_0 would be 400 MPa (Chawla, 1987).

Guided by the above estimates, we use the following values in the computations: $E_{11}/\sigma_0 = 335$, $\mu_{12}/\sigma_0 = 90$, $\mu_{23}/\sigma_0 = 75$, $K_{23}/\sigma_0 = 200$, $\nu_{12} = 0.27$, $F = 1.5$, $G = 0.5$ and $M = 8$. The constitutive parameters in eqn (113), above, are taken to be $\varepsilon_0 = 0.002$ and $n = 5$.

Four-node isoparametric elements with 2×2 Gauss integration and an independent

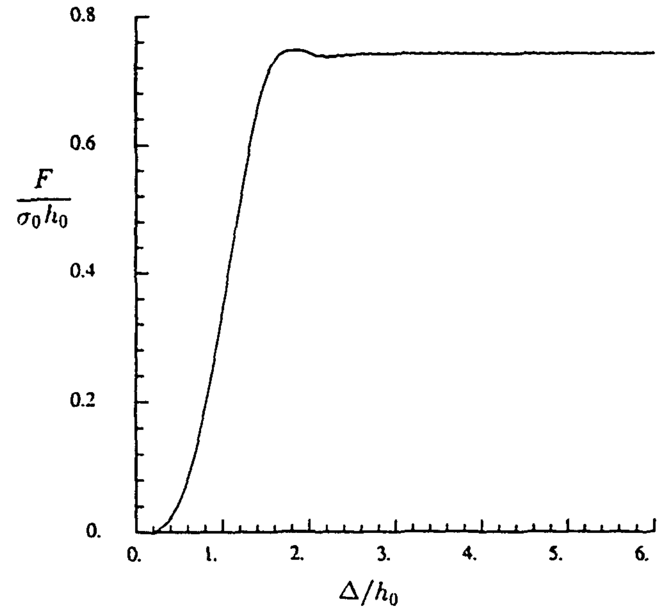

Fig. 6. The variation of the extrusion force.

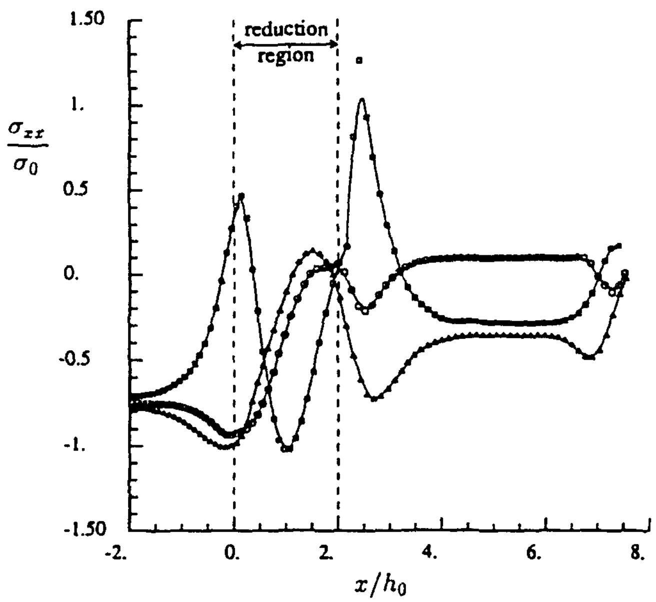

Fig. 7. The distribution of the longitudinal stress σ_{xx}.

interpolation for the dilatation rate are used in order to avoid artificial constraints on incompressible modes (Nagtegaal *et al.*, 1974).

Figure 5a shows the deformed finite element mesh and the extent of the active plastic elements at the end of the calculation; Fig. 5b shows the direction of transverse isotropy at the centroid of each element. The normalized extrusion force $F/(\sigma_0 h_0)$ is plotted versus the normalized piston displacement Δ/h_0 in Fig. 6. The x and y coordinates used in the two following figures are as shown in Fig. 5b, where the x-axis is the longitudinal axis of symmetry of the billet and $x = 0$ at the start of the reduction region of the die. Figure 7

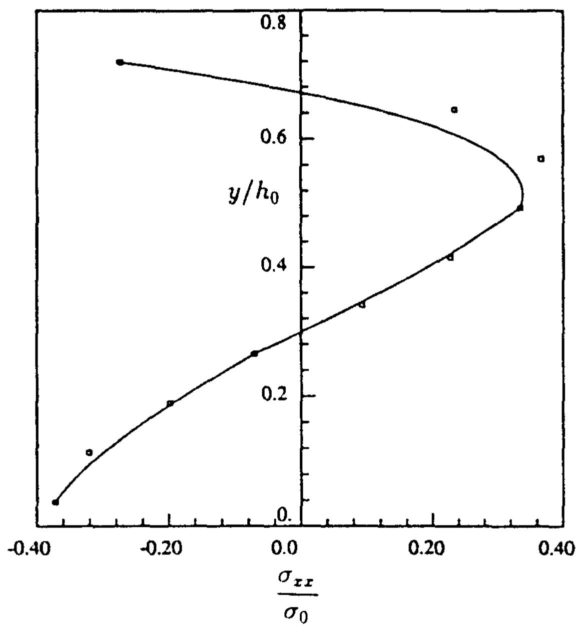

Fig. 8. Residual stress distribution across the section of the billet.

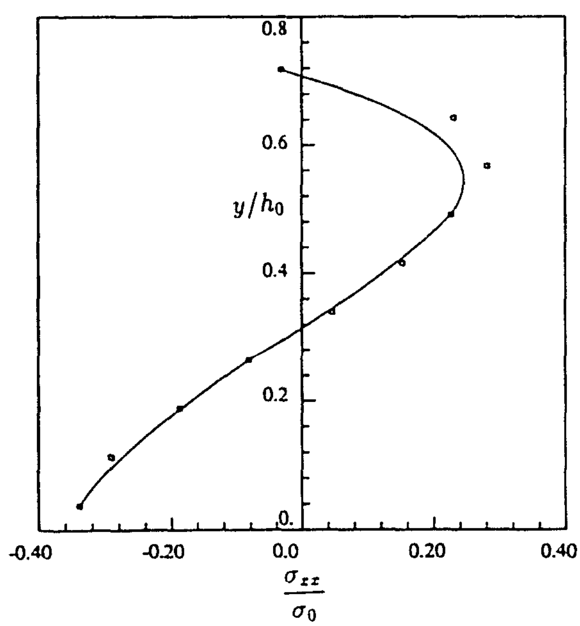

Fig. 9. Residual stress distribution across the section of the billet (isotropic elasticity).

shows the variation of the longitudinal stress σ_{xx} along lines passing through different rows of elements in the billet; in that figure, the open triangles, circles and squares correspond to the lower, fifth and top row of elements respectively. Figure 8 shows the variation of the residual longitudinal stress across the section of the billet after the exit from the die. It is interesting to note that the longitudinal residual stress is compressive near the outer surface of the billet; this contrasts with the results obtained for *isotropic* materials where the residual stress is found to be tensile near the free surface (Lee *et al.*, 1977).

A separate set of calculations is carried out, in which the same plastic parameters are used but the elastic response of the composite is assumed to be isotropic with $E/\sigma_0 = 300$ and $\nu = 0.3$. The resulting residual stresses across the section of the billet is shown in Fig. 9. The residual stress distribution is now close to zero near the surface, whereas near the axis of the billet is similar to that shown in Fig. 8. It appears, therefore, that the magnitude (and probably the sign) of the residual stress near the surface of the billet depends strongly on the elastic properties of the composite.

Acknowledgements—Fruitful discussions with Professor Y. F. Dafalias are gratefully acknowledged. The author would like to thank Dr R. M. Govindarajan and Mr J. Pereda for help with the computations. This work was carried out while the author was supported by the National Science Foundation through a Presidential Young Investigator Award (Grant MSM-8657860) and byALCOA (matching funds). Use of the computational facilities of the NSF MRL program at the University of Pennsylvania under Grant No. DMR-8818885 is gratefully acknowledged. The ABAQUS general purpose finite element program was made available under academic license from Hibbitt, Karlsson and Sorensen, Inc., Providence, RI.

REFERENCES

Aravas, N. (1991). Finite strain anisotropic plasticity: constitutive equations and computational issues. In *Advances in Finite Deformation Problems in Materials Processing and Structures* (Edited by N. Chandra and J. N. Reddy), AMD-Vol. 125, pp. 25–32. ASME, New York.

Aravas, N. and Aifantis, E. C. (1991). On the geometry of slip and spin in finite plastic deformation. *Int. J. Plasticity* **7**, 141–160.

Asaro, R. J. and Rice, J. R. (1977). Strain localization in ductile single crystals. *J. Mech. Phys. Solids* **25**, 309–338.

Boehler, J. P. (1975). Sur les formes invariantes dans le sous-groupe orthotrope de révolution des transformations orthogonales de la relation entre deux tenseurs symétriques du second order. *Z. Angew. Math. Mech.* **55**, 609–611.

Boehler, J. P. (1977). On irreducible representations for isotropic scalar functions. *Z. Angew. Math. Mech.* **57**, 323–327.

Boehler, J. P. (1979). A simple derivation of representations for non-polynomial constitutive equations in some cases of anisotropy. *Z. Angew. Math. Mech.* **59**, 157–167.

Chawla, K. K. (1987). *Composite Materials*. Springer, New York.

Christensen, R. M. (1979). *Mechanics of Composite Materials*. Wiley, New York.

Dafalias, Y. F. (1983). On the evolution of structure variables in anisotropic yield criteria at large plastic transformations. Colloque International du C.N.R.S. 351, sur Critères de Rupture des Matériaux à Structure interne Orientée (Edited by J. P. Boehler), Villard-de-Lans, France, June 1983 (to appear).

Dafalias, Y. F. (1984). The plastic spin concept and a simple illustration of its role infinite plastic transformations. *Mech. Mater.* **3**, 223–233.

Dafalias, Y. F. (1985a). The plastic spin. *J. Appl. Mech.* **52**, 865–871.

Dafalias, Y. F. (1985b). A missing link in the macroscopic constitutive formulation of large plastic deformations. In *Plasticity Today* (Edited by A. Sawczuk and G. Bianchi), pp. 135–151. Elsevier, Chichester, U.K.

Dafalias, Y. F. (1987a). Issues in constitutive formulation at large elastoplastic deformation, Part 1: Kinematics. *Acta Mech.* **69**, 119–138.

Dafalias, Y. F. (1987b). Anisotropy, reference configuration and residual stresses. In Constitutive Laws for Engineering Materials, Theory and Applications (Edited by C. S. Desai, E. Krempl, P. D. Kiousis and T. Kundu), Vol. 1, pp. 69–80. Elsevier, Amsterdam.

Dafalias, Y. F. (1988). Issues in constitutive formulation at large elastoplastic deformation, Part 2: Kinetics. *Acta Mech.* **73**, 121–146.

Dafalias, Y. F. and Rashid, M. M. (1989). The effect of plastic spin on anisotropic material behavior. *Int. J. Plasticity* **5**, 227–246.

Fardshisheh, F. and Onat, E. T. (1974). Representation of elastoplastic behavior by means of state variables. In *Problems of Plasticity* (Edited by A. Sawczuk), pp. 89–115. Noordhoff, Leyden.

Freund, L. B. (1970). Constitutive equations for elastic-plastic materials at finite strain. *Int. J. Solids Structures* **6**, 1193–1209.

Hahn, H. T. (1974). A finite-deformation theory of plasticity. *Int. J. Solids Structures* **10**, 111–121.

Halphen, B. (1975). Sur le champ des vitesses en thermoplasticité finie. *Int. J. Solids Structures* **11**, 947–960.

Hibbitt, H. D. (1984). ABAQUS/EPGEN—a general purpose finite element code with emphasis on nonlinear applications. *Nucl. Engng Des.* **77**, 271–297.

Hill, R. (1948). A theory of the yielding and plastic flow of anisotropic metals. *Proc. R. Soc.* **A193**, 281–297.

Hill, R. (1950). *The Mathematical Theory of Plasticity*. Oxford University Press, Oxford.

Hill, R. (1959). Some basic principles in the mechanics of solids without a natural time. *J. Mech. Phys. Solids* **7**, 209–225.
Hill, R. (1966). Generalized constitutive relations for incremental deformation of metal crystals by multislip. *J. Mech. Phys. Solids* **14**, 95–102.
Hill, R. (1967). The essential structure of constitutive laws for metal composites and polycrystals. *J. Mech. Phys. Solids* **15**, 79–95.
Hill, R. (1968). On constitutive inequalities for simple materials. *J. Mech. Phys. Solids* **16**, 229–242.
Hill, R. and Havner, K. S. (1982). Perspectives in the mechanics of elastoplastic crystals. *J. Mech. Phys. Solids* **30**, 5–22.
Hill, R. and Rice, J. R. (1972). Constitutive analysis of elastic–plastic crystals at arbitrary strain. *J. Mech. Phys. Solids* **20**, 401–413.
Hill, R. and Rice, J. R. (1973). Elastic potentials and the structure of inelastic constitutive laws. *SIAM J. Appl. Math.* **25**, 448–461.
Kratochvil, J. (1971). Finite-strain theory of crystalline elastic–inelastic materials. *J. Appl. Phys.* **42**, 1104–1108.
Kratochvil, J. (1973). On a finite strain theory of elastic–inelastic materials. *Acta Mech.* **16**, 127–142.
Lee, E. H. (1969). Elastic–plastic deformations at finite strains. *J. Appl. Mech.* **36**, 1–6.
Lee, E. H. (1981). Some comments on elastic–plastic analysis. *Int. J. Solids Structures* **17**, 859–872.
Lee, E. H. and Liu, D. T. (1967). Finite-strain elastic–plastic theory with application to plane wave analysis. *J. Appl. Phys.* **38**, 19–27.
Lee, E. H., Mallett, R. L. and Yang, W. H. (1977). Stress and deformation analysis of the metal extrusion process. *Comput. Methods Appl. Mech. Engng* **10**, 339–353.
Liu, I.-S. (1982). On representations of anisotropic invariants. *Int. J. Engng Sci.* **20**, 1099–1109.
Loret, B. (1983). On the effects of plastic rotation in the finite deformation of anisotropic elastoplastic materials. *Mech. Mater.* **2**, 287–304.
Lubliner, J. (1986). Normality rules in large-deformation plasticity. *Mech. Mater.* **5**, 29–34.
Lubliner, J. (1990). *Plasticity Theory*. Macmillan, New York.
Mandel, J. (1971a). Plasticité classique et viscoplasticité, Courses and Lectures, no. 97, International Center for Mechanical Science, Udine. Springer, Berlin.
Mandel, J. (1971b). Sur la décomposition d'une transformation élastoplastique. *C. R. Acad. Sci. Paris, Série A* **272**, 276–279.
Mandel, J. (1973). Equations constitutives et directeurs dans les milieux plastiques et viscoplastiques. *Int. J. Solids Structures* **9**, 725–740.
Mandel, J. (1974). Director vectors and constitutive equations for plastic and viscoplastic media. In *Problems of Plasticity* (Edited by A. Sawczuk), pp. 135–143. Noordhoff, Leyden.
Mandel, J. (1981). Sur la définition de la vitesse de déformation élastique et sa relation avec la vitesse de contrainte. *Int. J. Solids Structures* **17**, 873–878.
Mcmeeking, R. M. and Rice, J. R. (1975). Finite-element formulations for problems of large elastic plastic deformation. *Int. J. Solids Structures* **11**, 601–616.
Mohan, R., Ortiz, M. and Shih, C. F. (1992). An analysis of cracks in ductile single crystals: Part I—Anti-plane shear. *J. Mech. Phys. Solids* **40**, 291–313.
Moran, B., Ortiz, M. and Shih, C. F. (1990). Formulation of implicit finite element methods for multiplicative finite deformation plasticity. *Int. J. Numer. Methods Engng* **29**, 483–514.
Mulhern, J. F., Rogers, T. G. and Spencer, A. J. M. (1967). A continuum model for fibre-reinforced plastic materials. *Proc. R. Soc.* **A301**, 473–492.
Mulhern, J. F., Rogers, T. G. and Spencer, A. J. M. (1969). A continuum theory for a plastic elastic fibre-reinforced material. *Int. J. Engng Sci.* **7**, 129–152.
Nagtegaal, J. D. and De Jong, J. E. (1981). Some computational aspects of elastic–plastic large strain analysis. *Int. J. Numer. Meth. Engng* **17**, 15–41.
Nagtegaal, J. D., Parks, D. M. and Rice, J. R. (1974). On numerically accurate finite element solutions in the fully plastic range. *Comput. Methods Appl. Mech. Engng* **4**, 153–177.
Needleman, A. (1985). On finite element formulations for large elastic–plastic deformations. *Comput. struct.* **20**, 247–257.
Nemat-Nasser, S. (1982). On finite deformation elasto plasticity. *Int. J. Solids Structures* **18**, 857–872.
Ogden, R. W. (1984). *Non-linear Elastic Deformations*. Ellis Horwood, Chichester.
Rice, J. R. (1970). "A note on the 'small strain' formulation for elastic–plastic problems", Division of Engineering Technical Report No. N00014-67-A-0191-0003/81 to the Office of Naval Research, Brown University, Providence, RI.
Rice, J. R. (1971). Inelastic constitutive relations for solids: an internal-variable theory and its application to metal plasticity. *J. Mech. Phys. Solids* **19**, 433–455.
Sidoroff, F. and Teodosiu, C. (1986). Microstructure and phenomenological models for metals. In *Large Deformations of Solids: Physical Basis and Mathematical Modelling* (Edited by J. Gittus, J. Zarka and S. Nemat-Nasser), pp. 163–186. Elsevier, London, U.K.
Simo, J. C. (1985). On the computational significance of the intermediate configuration and hyperelastic stress relations in finite deformation elastoplasticity. *Mech. Mater.* **4**, 439–451.
Simo, J. C. and Ortiz, M. (1985). A unified approach to finite deformation elastoplastic analysis based on the use of hyperelastic constitutive equations. *Comput. Methods Appl. Mech. Engng* **49**, 221–245.
Smith, G. F. (1971). On isotropic functions of symmetric tensors, skew-symmetric tensors and vectors. *Int. J. Engng Sci.* **9**, 899–916.
Spencer, A. J. M. (1971). Theory of invariants. In *Continuum Physics* (Edited by A. C. Eringen), Vol. 1, pp. 239–253. Academic Press, New York.
Spencer, A. J. M. (1972). *Deformations of Fibre-Reinforced Materials*. Oxford University Press, Oxford.
Spencer, A. J. M. (1984). Constitutive theory for strongly anisotropic materials. In *Continuum Theory of the Mechanics of Fibre-Reinforced Composites* (Edited by A. J. M. Spencer), Courses and Lectures, No. 282, pp. 1–32. International Center for Mechanical Sciences, Udine. Springer, Berlin.

Teodosiu, C. and Sidoroff, F. (1976). A theory of finite elastoviscoplasticity of single crystals. *Int. J. Engng Sci.* **14**, 165–176.

Van der Giessen, E. (1989a). Continuum models for large deformation plasticity. Part I: Large deformation plasticity and the concept of a natural reference state. *Eur. J. Mech., A Solids* **8**, 15–34.

Van Der Giessen, E. (1989b). Continuum models for large deformation plasticity. Part II: A kinematic hardening model and the concept of a plastically induced orientation tensor. *Eur. J. Mech., A Solids* **8**, 89–108.

Walpole, L. J. (1981). Elastic behavior of composite materials: theoretical foundations. *Advances in Applied Mechanics* (Edited by C.-S. Yih), Vol. 21, pp. 169–242. Academic Press, New York.

Wang, C.-C. (1970a). A new representation theorem for isotropic functions: an answer to Professor G. F. Smith's criticism of my papers on representations of isotropic functions. Part 1. Scalar-valued isotropic functions. *Archs Ration. Mech. Anal.* **36**, 166–197.

Wang, C.-C. (1970b). A new representation theorem for isotropic functions: an answer to Professor G. F. Smith's criticism of my papers on representations of isotropic functions. Part 2. Vector-valued isotropic functions, symmetric tensor-valued isotropic functions, and skew-symmetric tensor-valued isotropic functions. *Archs Ration. Mech. Anal.* **36**, 198–223.

Weber, G. and Anand, L. (1990). Finite deformation constitutive equations and a time integration procedure for isotropic hyperelastic-viscoplastic solids. *Comput. Methods Appl. Mech. Engng* **79**, 173–202.

Willis, J. R. (1969). Some constitutive equations applicable to problems of large dynamic plastic deformation. *J. Mech. Phys. Solids* **17**, 359–369.